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EIGEN ENERGY EXCHANGE MODES OF MAGNETOSPHERE AT LONG-LASTING STOCHASTIC COUPLING WITH SOLAR WIND

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Abstract. Simulation of energy evolution for magnetosphere having infinite number of energy links with solar wind in assumption that all links are executed by energy continuity equation is conducted. Found closed form solution Υ is a function of magnetosphere's energy exchange rate y and in its physical meaning it is an integral efficiency of energy exchange. It is shown that quantity Υ is confined by the rate y. As a result, Υ and the total exchange energy E can take on certain discrete levels, as a result the spectrum for Υ and E is quantized.

1. Introduction

In terms of system science [1], a magnetosphere of the Earth (M_oE) is an open thermodynamic system (*OTS*) maintaining the permanent energy exchange with its environment where solar wind (*SW*) takes the main role. To highlight this fact, below we will call an environment of M_oE as *SW*.

Because the energy sources and sinks (energy agents) of M_oE are widely scattered in space and time, the research of an energy budget for M_oE causes serious issues. That is why usually the list of the accounted energy agents reduces to just a few ones which make the most significant contribution (based on experimental results and preliminary consideration) to M_oE dynamics.

Meanwhile, achievements of modern data science demonstrate an essential dependence of the obtained results on the quality of data [2]. In particular, it assumes consideration of contribution from all available data agents irrelevant of the prejudgement on its relative importance for the given problem. So, accounting of as many energy agents as possible becomes the necessity for accurate evaluation of energy budget of M_oE as well.

Also, considering the ever-changing character of agents it is scarcely possible to adequately describe M_oE in the terms of deterministic quantities. In this sense, use of a stochastic approach is regarded more warranted.

So, purpose of this report is to apply the model of energy evolution for stochastic system with infinite number of the random energy agents [3] to simulation of the energy spectrum of M_oE.

2. Approach substantiation

Follow [3], in the further text, we abstract from the real nature of the various energy fluxes between M_oE and *SW*. It is thought that the physical quantities involved into description of an energy exchange are not attributed to any specific process likewise electromagnetic, thermal, optical, mechanical and so on. Instead, as soon as any energy transportation which affects energy balance of M_oE upon arrival has occurred, it falls under our consideration.

Then, it is possible to unify all above energy exchange processes on the same mathematical basis. Use of an infinite number of energy links guarantees that every single exchange process will be accounted.

So, we take an energy continuity equation as a unified mathematical abstraction of a single energy link between M_oE and *SW*

$$\frac{\partial \varepsilon}{\partial t} = -Div\,\mathbf{J} \tag{1}$$

where ε is an energy volume density, t is time, $\boldsymbol{J}$ is energy flux, *Div* is a divergence operator. We consider an infinite multitude of similar independent links as an energy image of M_oE. And *SW* is simulated by an energy bath of unlimited capacity.

Though at every moment, all energy links are available for an energy transfer inwards or outwards M_oE, however only one link will be picking up and this link will take a role of the instant energy agent. It is believed that the random energy transfers δQ_{in} (inward M_oE) and δQ_{out} (outward M_oE) cannot coexist at the same time, *i.e.*, general energy flow $\delta Q = \delta Q_{in} \oplus \delta Q_{out}$, where $\oplus$ denotes exclusive disjunction.

3. Mathematical formalism

So, based on (1), concisely the mathematical model for energy evolution of M_oE is

$$\begin{cases} \dfrac{\partial \varepsilon_i}{\partial t} = -Div\,\mathbf{J}_i \end{cases} \tag{2}$$

which at counter $i \to \infty$ converts to

$$\frac{dU}{Q} = -\frac{dy}{y} x \,, \tag{3}$$

integration of (3) yields

$$\delta \Upsilon(x, y) = x \cdot \ln y \tag{4}$$

with the closed-form solution for an integral efficiency of energy exchange

$$\Upsilon(y) = -\iint_D \delta \Upsilon(x, y) = -\int_{-1}^{1} dx \cdot \int_0^y ln y \, dy$$

which after applying of boundary conditions is

$$\Upsilon(y) = y - y \ln y \tag{5}$$

and the total energy of exchange

$$E(y) = \int_0^y \Upsilon(z)\, dz = \frac{3y^2}{4} - \frac{y^2}{2} \ln y \tag{6}$$

where $x = \cos \beta$ is continuous random quantity, $y = J/J_0$ - unitless energy rate, U is internal system energy, normalizing constant $J_0 > 0$, $Q = J \cdot dS_A\, dt$, S_A is area, $\beta = d\boldsymbol{J} \wedge \boldsymbol{n}$, $\boldsymbol{n}$ is a unit normal oriented in the direction of incoming energy flux, $D \subseteq \boldsymbol{R}^2$ is phase space for all microstates of $\delta \Upsilon$.

Now, find y satisfying condition $\Upsilon(y) = 0$. Then, resolving (5) in respect of y

$$y_1 = \exp[1 + W_L(-1, -\frac{\Upsilon}{e})] = 0 \tag{7.a}$$

$$y_2 = \exp[1 + W_L(0, -\frac{\Upsilon}{e})] = e \tag{7.b}$$

where W_L is a Lambert function, $W_L(0)$ is the upper branch and $W_L(-1)$ is the lower one [4].

4. Boundary problem for function Υ

Solution $\Upsilon(y)$ is bounded within the range $[0, e]$ (7.a,b) and, formally, we have a two-point boundary value problem on the finite interval $[0, e]$

$$\Upsilon(0) = \Upsilon(e) = 0 \tag{8}$$

For (5), to meet (8) should hold

$$1 = \pm n \ln y_n \tag{9}$$

so, at $|\ln y| \le 1$, *i.e.*, in the y-range $[1/e, e]$

$$y_n = \exp[\pm \frac{1}{n}] \tag{10}$$

with appropriate non-trivial harmonics

$$\Upsilon_n = y_n - y_n \ln y_n \tag{11}$$

where $n = 1, 2, \ldots$.

So, the y-points (10) are the nodes of the discrete spectrum (Fig. 1).

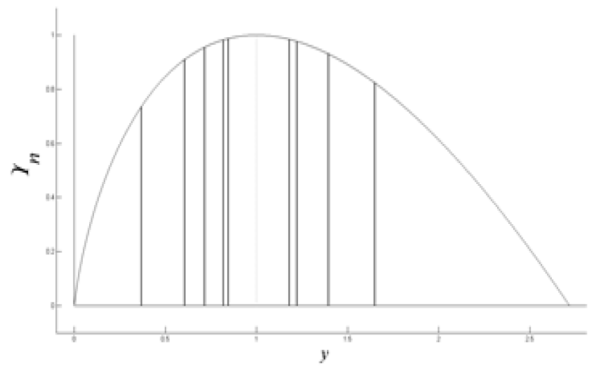

Figure 1. Discreteness of spectrum for integral efficiency of energy exchange Υ.
The discrete spectrum for Υ_n formed during evolution of *MoE* is shown. In the plot, by an abscissa axis, a unitless energy exchange rate $y = J/J_0$ is indicated, by an ordinate axis the integral efficiency of energy exchange Υ (a). A few first harmonics of Υ_n, are schematically shown by thick vertical segments in the range $1/e \le y \le e$. The harmonic $\Upsilon_n(y{=}e) = 0$, so it is not visible.

Note that formally, (6) allows to define its own discrete y-set for the total exchange energy E. However, that y-set does not have so remarkable features as the y-set for Υ (10) does [3], so we do not consider discretization of (6) here.

5. Results and discussion

The first thing to highlight here is that the found y-discreteness of M_oE calls to the rate of an energy exchange y. We mean that it does not appeal to the well-known magnetospheric cavity modes and field-line resonances usually appearing in the Pc5 range [5] or the modes caused by the changes of dynamic pressure in *SW* [6]. Instead, it comes from the limitation imposed on the y-range with the positive value of energy efficiency Υ. Consequently, we could conclude that y-discreteness of M_oE emerges as a result of an energy scaling of an energy exchange process. The scaling norm is the one full lifetime cycle (positive Υ) with the with the y-length equal to e. In other words, scaling in [5] stems from the limitation of the *state* (for ex. magnetosphere cavity), but discovered scaling stems from the limitation of the *process* (energy exchange).

During the first y-segment with continuous spectrum ($y \leq OP$), M_oE accumulates an energy and all modes of an energy exchange have the same priority. However, if the rate y achieves *OP*, situation changes, and some (infinite but counted) energy exchange modes receive the higher priority. As an example of such high priority energy exchange mode could be a magnetosphere substorm which typically follows the time period of the less or more quiet (energy accumulation stage) magnetospheric conditions [7].

It is worth to note that the discrete spectrum at $OP < y < TS$ still keeps some noise-like background, *i.e.*, actually it is a *quasi-discrete* one. It looks reasonable as the bounds (7.a) and (7.b) are not tough. Actually, at these points, efficiency of energy exchange $\Upsilon(y)$ changes its sign from negative to positive and vice versa. Hence, the bounds (7.a) and (7.b) are rather the singular points where the qualitative leap in M_oE functioning happens.

On the other hand, the quasi-discrete form of spectrum means that the backstage physical mechanism dominating while in the continuous spectrum does not totally disappear, it is still there but its influence has reduced. So, the discrete and continuous segments of spectrum overlap in some extent. As a result, all above said leads to the not perfect discrete pattern.

Note that the meaning of the y-segment with the continuous spectrum is twofold. Firstly ($0 < y < OP$), it is to supply energy of random variations in an amount sufficient to support the more advanced and complicate discrete segment on the qualitatively different level of *MoE* operation. Secondly ($TS < y < TP$), is to provide suitable dissipation of the accumulated energy before completely flushing it out.

Compare the y-dependence of efficiency Υ (Fig. 1) and total exchange energy E (Fig. 2 (a)). We see that visually the discrete spectrum for Υ ends much earlier (at $y = e^{1/2}$) versus the spectrum for E (at $y = e$). However, it is accurate result as $\Upsilon(y = e) = 0$ and the last visible harmonic is Υ (at $y = e^{1/2}$) in contrast to the total energy E as it is of zero only at the one point $y = 0$. So, we conclude that the quantization of disappears once Υ becomes negative.

Note that from (10) the "condensing" of the spectral lines E_n at $y \rightarrow 1$ should be observed (Fig. 2.a). It is agreed with the meaning of SP $(y = 1)$ as a stationarity point where energy exchange acquires the smoother character without the drastic changes between individual energy harmonics.

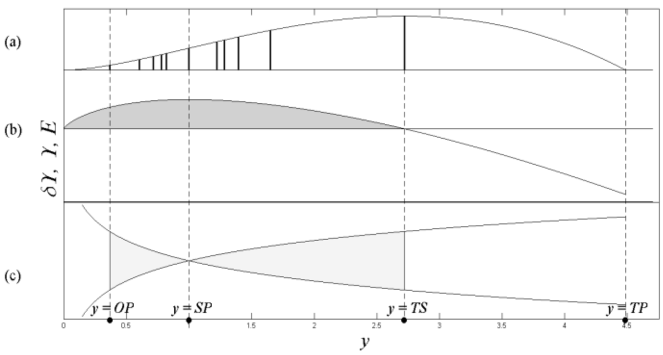

Figure 2. Solution for an energy evolution of the *MoE* – total exchange energy *E* (a), integral efficiency of energy exchange Υ (b), instantaneous efficiency of an energy exchange $\delta\Upsilon$ (c).

In the plot, by an abscissa axis a unitless energy rate $y = J/J_0$ is indicated. The quasi-quantized spectrum is between the points *OP* and *TS*, the range with positive Υ is between the points *0* and *TS.* The quasi-discrete energy levels E_n are shown by the thick vertical segments in the upper panel (a). It is seen that levels E_n tend to "condense" around the stationarity point *SP*. The point *TS* marks termination of the quasi-quantized spectrum, *TP* – termination of the whole *MoE*'s energy evolution. Phase space *D* for all possible microstates of $\delta\Upsilon$ is shown in the light grey, area of positive Υ in the dark grey.

Summarizing, based on the model of *OTS* with an infinite number of energy agents, we investigated evolution of energy spectrum of *MoE* and discovered presence of an interleaving between the continuous and quasi-quantized form of a spectrum. Such theoretical effect could be useful in explanation of some global impulse-like processes in Earth magnetosphere.

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