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THEORETICAL MODEL OF STEADY-STATE MAGNETIC RECONNECTION IN THE ELECTRON HALL MAGNETOHYDRODYNAMICS APPROXIMATION

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Abstract

Theoretical model of steady-state magnetic reconnection in an infinite current layer in incompressible, collisionless, nonresistive plasma, except of the electron diffusion region, is developed. The model is built using the electron Hall magnetohydrodynamics approximation. Solution structure is determined by the Grad-Shafranov equation for the magnetic field potential. The developed model demonstrates all essential Hall reconnection features, namely protons acceleration up to the Alfven velocities, and Hall current system and magnetic field structure forming. The model allows claiming that the necessary condition of the steady-state reconnection to exist is the electric field potential jump across electron diffusion region and separatrices. The magnitude of this jump must be proportional to the external magnetic pressure. Besides the electron velocity has to grow up to the electron Alfven velocity inside the diffusion region and on the separatrices and this is the necessary condition as well.

Introduction

Numeric experiment and in-situ observation data obtained over the last years permit considering the Hall MHD (HMHD) approximation as a minimal sufficient description for the magnetic reconnection process in collisionless plasma, [e.g. 1,2,3]. The Hall effect breaks the frozen-in condition for protons therefore protons become insensitive to the magnetic field. But this effect keeps the magnetic field frozen in the electron fluid [4]. It is well known that a small region around the *X*-point exists, where the electron Hall MHD (eHMHD) approximation is permitted to be used [5]. Indeed, the proton's velocities are negligibly small compared to the electron's ones in the nearest vicinity of the stagnation *X*-point. Hence one may consider the electric current in this region as electron current only. This assumption is the matter of eHMHD. The applicability domain of this approach is smaller than the HMHD region, its size is not bigger than the proton inertial length. Nevertheless the eHMHD approximation, being simpler then HMHD, is proved to be of material significance. Namely it allows building a rich in content analytical model of steady-state reconnection based on the Grad-Shafranov equation (Gr.-Sh. eq.) solution.

The problem formulation

We examine the problem of steady-state magnetic reconnection in an infinite current layer. We intend to build the analytical model of this process in the nearest vicinity of the X-line using the eHMHD approximation, so the modeling region size is bounded by proton inertia length (in the direction of proton acceleration). As for the electron diffusion region (EDR) our aim is to avoid its internal processes description, only its size is considered. Outside this region the plasma is supposed to be nonresistive. Besides we suppose the plasma to be electro-neutral and incompressible and electrons to be cold ($T_e << T_p$). The last condition is usual in the numeric modeling, see [e.g. 6]. It allows neglecting of the electron pressure comparatively to the proton one. The magnetic field structure is shown schematically in Fig.1. The coordinate system choice is fixed in this figure:

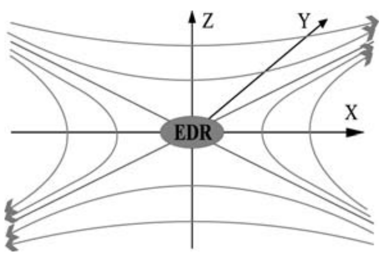

Fig.1. Scheme of the magnetic field structure

X-axis is field-at-infinity-aligned, *Y*-axis coincides with the electric current direction and *Z*-axis is perpendicular to them both (Fig.1).

The mathematical formulation of the problem consists of seven equations:

$$\rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + \frac{1}{c}\mathbf{j}\times\mathbf{B}, \tag{1}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} = \frac{1}{nec}\mathbf{j}\times\mathbf{B}, \tag{2}$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}, \tag{3}$$

$$\nabla\times\mathbf{E} = 0, \tag{4}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{5}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{6}$$

$$\mathbf{j} \cong -ne\mathbf{V_e}. \tag{7}$$

The last equation here is eHMHD approximation where $\mathbf{V_e}$ designates electron bulk velocity while $\mathbf{V}$ is plasma bulk velocity. Note that the generalized Ohm's Law (2) in nonresistive HMHD may be rewritten in a form of an electron frozen-in condition:

$$\mathbf{E} + \frac{1}{c}\mathbf{V_e} \times \mathbf{B} = 0. \qquad (8)$$

Since the infinite length of the current layer all variables in these equations are functions of two spaces coordinates x and z only while we have chosen the current direction as y axis. Under this condition and assuming a steady-state case Faraday's Law (4) leads to the E_y constancy. So we define:

$$E_y = \varepsilon E_A, \qquad (9)$$

where ε is reconnection rate and E_A is the Alfvén electric field, $E_A=(1/c)B_0V_A$. Here B_0 is the magnetic field value at infinity and V_A is the corresponding proton Alfvén velocity.

Grad-Shafranov equation

Here we rearrange the original system (1) – (7) to the system of four recurrent equations based on the Gr.-Sh. eq. First we trespass to dimensionless units. We normalize the magnetic field strength by the external field value B_0, the proton and electron bulk velocity (V_p and V_e) by the proton Alfven velocity value V_A, the electric field strength by the Alfven value E_A, and space scales by l_p – the proton inertial length value. l_p is defined as the light velocity c divided by the proton plasma frequency ω_p:

$$l_p = \frac{c}{\omega_p} = c\sqrt{\frac{m_p}{4\pi n e^2}}\,. \qquad (10)$$

We introduce potentials of the electric and magnetic field (φ and A) and flow-function for the electron bulk velocity Ψ, following the common way [5]:

$$\mathbf{E}_\perp = -\nabla_\perp \varphi, \qquad (11)$$

$$\mathbf{B}_\perp = \nabla \times (A\mathbf{e_y}), \qquad (12)$$

$$\mathbf{V}_{\mathbf{e}\perp} = -\nabla \times (\Psi \mathbf{e_y}). \qquad (13)$$

Here and below symbol ⊥ designates the xz plane that is perpendicular to the current direction defined by the unit vector $\boldsymbol{e_y}$. Then we take into consideration equation of motion (1). Expressing the current density $\boldsymbol{j}$ through the electron velocity $\boldsymbol{V_e}$ according to the eHMHD approximation (7) we obtain from (8):

$$\mathbf{j} \times \mathbf{B} = -\mathbf{V}_e \times \mathbf{B} = \mathbf{E} = -\nabla \varphi. \qquad (14)$$

Substituting the result in the equation (1) and replacing the plasma bulk velocity $\boldsymbol{V}$ by the proton velocity $\boldsymbol{V_p}$ (neglecting of terms of m_e/m_p order), we obtain Bernoulli's Law for the proton motion:

$$\frac{1}{2}V_{p\perp}^2 + p + \varphi = const_{traj}, \qquad (15)$$

where $const_{traj}$ denotes the constant along trajectory. Equations (3) – (6) then take form:

$$\nabla \varphi = \mathbf{V}_e \times \mathbf{B}, \qquad (16)$$

$$\nabla \times \mathbf{B} = -\mathbf{V_e}, \qquad (17)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (18)$$

$$\nabla \cdot \mathbf{V_p} = 0. \qquad (19)$$

Expressing functions $\boldsymbol{B}_\perp$ and $\boldsymbol{Ve}_\perp$ through the potentials A and Ψ according to (12, 13) and substituting result to the equation (17) we obtain significant identities:

$$V_{ey} \equiv \mathbf{\Delta} A, \qquad B_y \equiv \Psi. \qquad (20)$$

Further we consider equation (16). Scalar products of this equation and vectors $\boldsymbol{B}$ and $\boldsymbol{V_e}$, respectively, can be written as follows:

$$(\mathbf{B}_\perp \cdot \nabla_\perp)\varphi \equiv \frac{\partial \varphi}{\partial \mathbf{B}_\perp} = \varepsilon B_y, \qquad (21)$$

$$(\mathbf{V}_{\mathbf{e}\perp} \cdot \nabla_\perp)\varphi \equiv \frac{\partial \varphi}{\partial \mathbf{V}_{\mathbf{e}\perp}} = \varepsilon V_{ey}. \qquad (22)$$

On the other hand the y-component of equation (16) is precisely a Jacobian of the transformation of variables $(x,z) \rightarrow (A,\Psi)$ while A and Ψ are defined by equations (12, 13):

$$\varepsilon = V_{ez}B_x - V_{ex}B_z \equiv \left|\frac{\partial(A,\Psi)}{\partial(x,z)}\right| \qquad (23)$$

In the new variables operators $\partial/\partial\boldsymbol{B}_\perp$ and $\partial/\partial\boldsymbol{Ve}_\perp$ take form:

$$\frac{\partial}{\partial \mathbf{B}_\perp} \equiv \varepsilon \frac{\partial}{\partial \Psi}, \qquad \frac{\partial}{\partial \mathbf{V}_{\mathbf{e}\perp}} \equiv \varepsilon \frac{\partial}{\partial A}. \qquad (24)$$

Considering identities (20, 24) equations (21, 22) should be written as follows:

$$\varepsilon \frac{\partial \varphi}{\partial \Psi} = \varepsilon \Psi, \qquad \varepsilon \frac{\partial \varphi}{\partial A} = \varepsilon \mathbf{\Delta} A. \qquad (25,26)$$

Integrating the first of these equations we obtain equation for electric potential φ:

$$\varphi = \frac{1}{2}\Psi^2 + G(A). \qquad (27)$$

Operator $\partial/\partial\boldsymbol{B}_\perp$ being applied to the function Ψ gives (according to (24)):

$$\frac{\partial \Psi}{\partial \mathbf{B}_\perp} = \varepsilon \frac{\partial \Psi}{\partial \Psi} = \varepsilon. \qquad (28)$$

So if we know potential A, we can find the magnetic field components B_x and B_z and then potential $\Psi \equiv B_y$ according to equation:

$$\frac{\partial A}{\partial x}\frac{\partial \Psi}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial \Psi}{\partial x} = \varepsilon. \qquad (29)$$

At last expressing potential φ according to (27) and substituting the result in the equation (26) we obtain the Grad-Shafranov equation for the potential A:

$$\mathbf{\Delta} A = \frac{dG(A)}{dA}. \qquad (30)$$

Thus, four recurrent equations formulate our problem:

$$\begin{aligned} &\frac{1}{2}V_{p\perp}^2 + p + \varphi = const_{traj}, \\ &\varphi = \frac{1}{2}\Psi^2 + G(A), \\ &\frac{\partial A}{\partial x}\frac{\partial \Psi}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial \Psi}{\partial x} = \varepsilon, \\ &\mathbf{\Delta} A = \frac{dG(A)}{dA}. \end{aligned} \qquad (31)$$

Now scheme for the system (31) solution finding is evident. By solving the Gr.-Sh. eq. we define potential A. Then we find magnetic field $\mathbf{B}_\perp$ and Ψ which is equal to B_y. After that the potential φ is evaluated. The last unknown function we need for the proton motion finding is the gas pressure p. This problem may be passed over by the boundary layer approximation (see below). Besides the Gr.-Sh. eq. being complicated in itself contains the unknown function $G(A)$. Fortunately this function is unknown but not unimaginable.

Function G(A)

Function $G(A)$ is defined by the physical conditions of the problem. To make it out we first note that according to the magnetic field geometry (see Fig.1) function $A(x,z)$ has a saddle-type configuration, so we can simply define the sign of function $A(x,z)$. Its sign is positive in the outflow regions (OR), negative in the inflow regions (IR) so $A(x,z)=0$ on the separatrices (in the origin including). Then we put it by and consider function V_{ey} that is dG/dA. Due to the frozen-in condition (8) which is satisfied everywhere except EDR we can claim that the acceleration of electrons in the y direction is unrealizable out of EDR. Inside EDR electrons tear off the magnetic field and the electric field E_y accelerates them unimpeded. Hence, in the origin point function V_{ey} has minimum (because the electric current direction coincides with the positive direction of the y axis). And sharpness of this minimum depends on the EDR size. Formally the approximated Ohm's Law we used here (2) implies the EDR absence and leads to the δ-function type of V_{ey}. Replacing δ-function by any function with non-zero peak width we therethrough define the EDR size. Composing information about function A and $V_{ey}\equiv dG/dA$ we can figure out the likely behavior of the function dG/dA. By integrating this function we obtain $G(A)$ (see Fig.2).

The boundary layer approximation

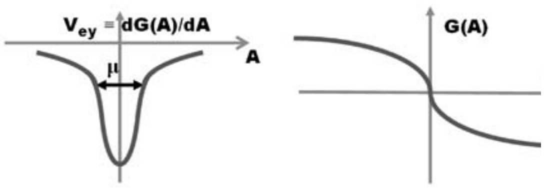

Fig.2. Functions V_{ey} and $G(A)$. μ is the EDR size in the A-space.

Geometry of the problem permits the boundary layer approximation usage. Namely the separatrices incline angle tangent is the ε order value. As far as $\varepsilon<<1$, the derivatives in x-direction are much less then in z-direction and x-components of all quantities are much bigger then z-components. In particular it means that $V_{pz}<<V_{px}$, so $V_p^2 \approx V_{px}^2$ (here V_p designates the proton velocity in the xz-plane). Furthermore rewriting equation (1) as follows:

$$\rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla\left(p+\frac{B^2}{8\pi}\right)+\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}, \qquad (32)$$

under this scaling reason we conclude that the full pressure $P=p+(1/8\pi)B^2$ is constant along z-direction. Therefore we have (in dimensionless units):

$$\frac{\partial p}{\partial z} = -\frac{1}{2}\frac{\partial B^2}{\partial z}. \qquad (33)$$

Recalling that $B_y\equiv\Psi$, under these conditions we obtain from the equations (31.1, 31.2):

$$\frac{\partial}{\partial z}\left[\frac{1}{2}V_{px}^2 - \frac{1}{2}(B_x^2+B_z^2)+G(A)\right]=0. \qquad (34)$$

This equation and incompressibility condition (6) allow the proton motion evaluation, if function $G(A)$ is known and magnetic field is found.

At last neglecting the x-derivative in the Gr.-Sh. eq. (31.4) and integrating it once by z we obtain the boundary problem for potential A:

$$\frac{\partial A}{\partial z} = \mp\sqrt{2}\cdot\sqrt{G(A)+C(x)}, \qquad (35)$$
$$A|_{z=0} = G^{-1}[-C(x)],$$

where $C(x)$ is defined by the magnetic pressure at infinity and G^{-1} is the inverse function for G. Thus the boundary layer approximation completes the analytical model of our problem construction.

Results

The problem formulation based on the Grad-Shafranov equation (31) allowed obtaining some impotent results without finding this system solution. This is the appreciable advantage of the approach that has been developed.

The significant role of B_y component is established. It turns out that B_y is the electron bulk velocity flux-function (13, 20). It means that curves $B_y=const$ are the electron trajectories in the xz plane (see Fig.4).

It is found that an electric field potential jump across the EDR and separatrices is the imperative feature of steady-state reconnection. The magnitude of this jump is determined by the constraints for equation (35). According to the nonnegativity condition of the radicand $maxG(A)$ must be equal to $1/2$ so jump of $G(A)$ is equal to 1 (see Fig.2). As far as magnetic field component $B_y\equiv\Psi<1$ equation (31.2) claims that an electric field potential jump must be proportional to the external magnetic pressure as well: $\Delta\varphi\sim\Delta G(A)\sim 1$, or in dimensional units:

$$\Delta\varphi \sim \frac{B_0^2}{4\pi ne} \qquad (36)$$

Under this condition the proton velocity reaches up to the Alfven value in the OR. Indeed we may rewrite Bernoulli's Law for protons (34) as follows:

$$\frac{1}{2}V_{px}^2 + P - \frac{1}{2}(B_x^2+B_z^2)+G(A) = const_{traj}. \qquad (37)$$

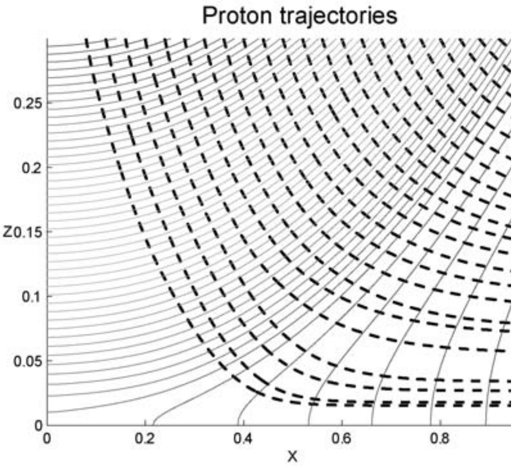

Fig.3: Proton trajectories (dashed) and magnetic field structure (solid).

The full pressure dependence on x-coordinate as well as external magnetic field dependence is the ε-order variation. At contrary function *G(A)* tends to its maximum in the IR (*1/2*) and to its minimum in the OR (*-1/2*). Magnetic field component V_x is equal to *1* approximately in the IR and it is ε-order value in OR. V_z component is small everywhere. Equation (37) under these simple consideration leads to the announced conclusion.

As a consequence the powerful mechanism of electron acceleration in the direction of the *X*-line is required. It is to accelerate electrons up to the electron Alfven velocity value inside the diffusion region and on the separatrices. This estimation is obtained as follows: (38)

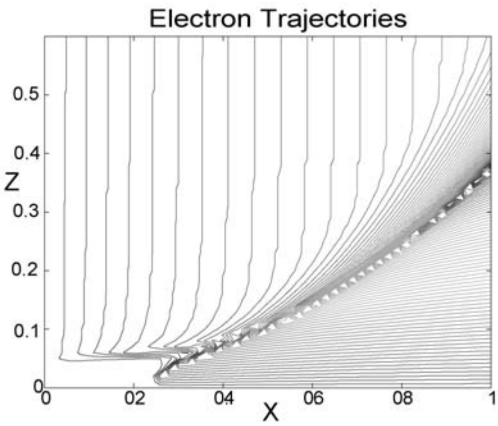

Fig.4: electron trajectories plotted as equipotential lines of B_y. Empty rectangle near the origin is the EDR where our numeric scheme is not applicable.

$$\Delta G(A) = \int_0^{l_e/l_p} \frac{dG(A)}{dA}\frac{\partial A}{\partial z}dz \sim \int_0^{l_e/l_p} V_{ey} B_x dz \sim V_{ey}\frac{l_e}{l_p} \sim 1$$

where l_e is the electron inertial length and l_e/l_p is the cross sectional size of EDR measured in proton inertial length units. Thus we obtain:

$$V_{ey} \sim \frac{l_p}{l_e} = \sqrt{\frac{m_p}{m_e}} \qquad (39)$$

Numerical solution was built as well. The recurrent system (31) was solved in the boundary layer approximation. It means that approximated equation (35) was considered instead of the precise equation (31.4). The modelling function *G(A)* was chosen as follows:

$$G(A) = -\frac{1}{\pi}\arctan\frac{A}{\mu}, \qquad (40)$$

where μ is the EDR size in the *A*-space. The boundary condition was set as function $B_z(x,0)$. This function must be equal to zero in the origin due to symmetry and it must trend to the constant out of EDR when electrons become frozen-in due to the Ohm's Law (8). We chose this function as follows:

$$B_z(x,0) = \chi(1-\exp(-\alpha\sqrt{x})). \qquad (41)$$

Here we present two results demonstrating the obtained solution for the fixed parameter values: $\mu=0.0075$, $\alpha=0.5$, $\chi=0.2$, and $\varepsilon=0.3$.

The numerical solution completely confirms the analytical conclusions. Solution demonstrates all essential Hall reconnection features, namely the proton acceleration, the forming of Hall current system (Figs 3,4) and the quadrupole magnetic field structure. These features were pointed out by Sonnerup [7] and observed by a number of authors [e.g. 2, 3].

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