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NON-LINEAR LANDAU DAMPING DUE TO GENERATION OF SECONDARY WAVES

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Abstract. In the paper a particular solution for non-linear Landau damping typical for waves of large amplitude is considered. We examine the electric charge density associated with the plateau. The charge density appears to be $\rho = -e f_{pl} 2v_1$, where $f_{pl} = f(v = v_{ph})$, and has harmonics of the form $\cos[m k (x - v_{ph} t)]$, where m is the number of a harmonics. These harmonics can generate secondary waves with higher wave numbers than those in the primary wave. The transmission of energy into the secondary waves leads to extra damping of the primary wave. The amplitude of the electric field of the wave of the second harmonics is estimated. It is considered that the contribution of higher harmonics is insignificant. Also, we estimate the efficiency of the mechanism by comparing the energy spent on generation of secondary waves with that spent on the formation of the plateau.

1. Introduction

Damping in time or in space of small-amplitude plasma waves in a collisionless plasma was predicted by Landau in 1946 [1]. In his perturbation theory, the electric field of the wave is assumed to be small, and the collisionless Boltzmann equation is linearized by neglecting $\partial f_1/\partial v$ compared to $\partial f_0/\partial v$ (where f_0 and f_1 are the equilibrium and perturbed parts of electron velocity distribution function, respectively). The linear theory is not applicable to large-amplitude waves. For the waves of large amplitude it is very difficult to obtain strict solutions, only the particular solutions being accessible. For sinusoidal waves this problem was addressed by O'Neil [2]. The experimental check of Landau damping was performed by Malberg and Warton in 1965 [3], and later by Defler and Simonen [4].

2. The basic equations

The propagation of plasma waves is described by the set of two equations including the Vlasov equation and Poisson equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial v} = 0 \ , \qquad \frac{\partial^2 \varphi}{\partial x^2} = 4\pi e\left(\int_{-\infty}^{+\infty} f\, dv - n_0\right) \tag{1}$$

where f – velocity distribution function, φ - electric potential, e – the charge of electron, m – the mass of electron, n_0 the unperturbed concentration.

3. Linear approach

By linearizing equations (1), Landau obtained the following dispersion relation:

$$1 - \frac{4\pi e^2}{k^2 m}\int_{-\infty}^{\infty} \frac{P}{v - v_{ph}}\frac{\partial f_0}{\partial v}\, dv - \frac{4\pi e^2}{km}\frac{\pi i}{|k|}\left.\frac{\partial f_0}{\partial v}\right|_{v=v_{ph}} = 0 \tag{2}$$

where P stands for the principal value. The real part of equation (2) describes the wave propagation. The imaginary part indicates either damping of the wave (for $\partial f_0/\partial v|_{v=\omega/k} < 0$), or its growth (for $\partial f_0/\partial v|_{v=\omega/k} > 0$). The presence of an imaginary part in equation (2) is connected with the resonance electrons, moving with the velocities close to the phase velocity of the wave [5].

4. Physics of damping

The physical meaning of Landau damping can be understood in terms of energy exchange between the resonance electrons and the wave. For the description of the physical sense of damping we will take an advantage of the Maxwellian distribution function. The evolution of the Maxwellian distribution under energy exchange between the wave and resonance electrons in a potential hole of the wave can be seen in Fig. 1 (a, b, c). In the first half-period of fluctuations the wave gives energy to resonance particles, and thus its amplitude decreases. At this time the derivative of the Maxwellian distribution function over velocity is negative ($\partial f_0/\partial v|_{v=\omega/k} < 0$) in the vicinity of the phase velocity of the wave $v = v_{ph}$. In this period some of the resonance particles can become untrapped because the potential barrier is decreased. In the next half-period the resonance particles give a part of their energy back to the wave, thus the amplitude of the wave increases but it does not reach the initial level, as the number of the resonance particles became smaller. At that time, the derivative of the distribution function is positive ($\partial f_0/\partial v|_{v=\omega/k} > 0$), i.e. the number of fast electrons is increased. An increase in the wave energy corresponds to instability. Such fluctuations of the distribution function will proceed till a plateau forms in the vicinity of $v = v_{ph}$. The wave will fade because in the Maxwellian distribution function the number of slow electrons, taking energy from the wave, exceeds that of fast electrons, giving the energy back to the wave [6, 7].

5. Estimate of damping

After plateau formation, the total increase in particle energy will be

$$\delta W = \int_{v_{ph}-\Delta v}^{v_{ph}+\Delta v} (f_0 - f)\frac{mv^2}{2}dv \tag{3}$$

where f_0 and f are the initial and final distribution functions, respectively; Δv is the half-width of the plateau determined from the expression

$$\Delta v = \frac{e\varphi_0}{mv_{ph}} = \frac{eE_0}{kmv_{ph}}, \tag{4}$$

where φ_0 and E_0 are the peak values of the potential and electrical field, respectively.

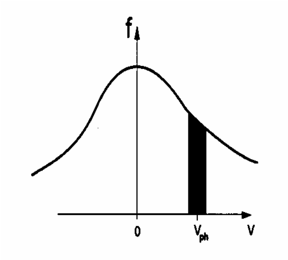

Fig.1, a. Maxwellian distribution function at t=0

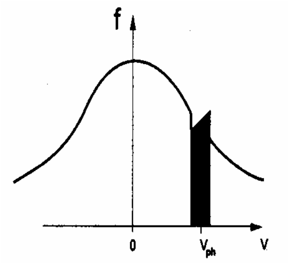

Fig.1, b. Maxwellian distribution function at $t = Tosc/2$.

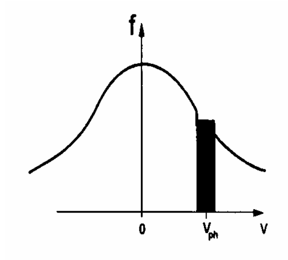

Fig. 1, c. Formation of plateau on distribution function in the vicinity of $v = v_{ph}$

Initial distribution function in the vicinity of the velocity v_{ph} can be presented as

$$f = f_0 + \frac{\partial f}{dv}(v - v_{ph}) \tag{5}$$

Substituting (4) and (5) into (3), we obtain

$$\delta W = -\frac{mv_{ph}^2}{2}\Delta v^2 \left.\frac{\partial f}{\partial v}\right|_{v=v_{ph}} = -\frac{e^2E_0^2}{2k^2m}\left.\frac{\partial f}{\partial v}\right|_{v=v_{ph}} \tag{6}$$

The density of the wave energy is equal to

$$W_w = \frac{\langle E^2 \rangle}{8\pi}\frac{d(\omega\varepsilon)}{d\omega} \approx \frac{E_0^2}{16\pi}\times 2 = \frac{E_0^2}{8\pi} \tag{7}$$

Energy (6) given to the electrons is taken from the wave energy. Hence, the amount of the wave energy spent on heating of the electrons is equal to

$$\frac{\delta W}{W_w} = -\frac{4\pi e^2}{k^2m}\left.\frac{\partial f}{\partial v}\right|_{v=v_{ph}} \tag{8}$$

The Maxwellian distribution function has the form

$$f(v) = \frac{n_0}{\sqrt{\pi}v_T}\exp(-v^2/v_T^2) , \tag{9}$$

where n_0 is the unperturbed concentration, v_T the thermal speed of electrons. Substituting (9) into (8), we obtain

$$\frac{\delta W}{W_w} = \frac{2}{\sqrt{\pi}}\frac{\omega_0^2 v_{ph}}{k^2v_T^3}\exp(-v_{ph}^2/v_T^2) , \tag{10}$$

where $\omega_0 = (4\pi e^2 n_0/m)^{1/2}$ is the plasma frequency. Adopting $v_{ph} \approx \omega_0/k$, we obtain finally

$$\frac{\delta W}{W_w} = \frac{2}{\sqrt{\pi}}\frac{v_{ph}^3}{v_T^3}\exp(-v_{ph}^2/v_T^2). \tag{11}$$

From expression (11) one can see that at $v_{ph} >> v_T$ only a small part of the wave energy is given to electrons.

6. Non-linear interaction

The linear approach is fair only for waves of small amplitude, for waves of large amplitude the non-linear damping takes place. The wave of large amplitude does not fade exponentially, as the linear theory predicts. Instead, a decrease of the amplitude is observed, then, again, a growth, and eventually the wave becomes stationary. Such a behavior of the wave is well consistent with the picture of particle capture considered in section 4. At the oscillations of the resonance electrons in the potential hole of the wave, there will be generated secondary plasma waves. In the course of energy exchange with the wave, electrons, as a whole, will take energy from the wave, the part of this energy being spent on secondary waves. The energy is obtained from the primary wave, which thus

fades. There is a possibility of non-linear transforming of primary wave energy to the energy of the secondary waves. Let us estimate the efficiency of such a transforming, considering the primary wave to be monochromatic.
The solution of equation (1) is any function of the integrals of movement. For a wave, propagating with the phase velocity of v_{ph}, the integral of movement is the total energy of electrons in the coordinate system of the wave.
Then

$$f = f(I) \tag{12}$$

where the integral of movement I can be presented as

$$I = \sqrt{(v - v_{ph})^2 - \frac{2e}{m}(\varphi + \varphi_0)}\ . \tag{13}$$

where φ_0 is the peak value of the potential. This can be illustrated with the Maxwellian distribution function. If one considers that for $\varphi + \varphi_0 = 0$ the distribution function has the form of (9), then, combining (12) and (13), we obtain the distribution function for any φ.

$$f(v,\varphi) = \frac{n_0}{\sqrt{\pi} v_T} \exp(-(v_{ph} \pm \sqrt{(v - v_{ph})^2 - \frac{2e}{m}(\varphi + \varphi_0)})^2 / v_T^2)\ . \tag{14}$$

The upper sign before the square root corresponds to $v > v_{ph}$, the lower sign to $v < v_{ph}$. If the value under the square root appears to be negative, it should be taken zero. Under these conditions, formula (14) describes the distribution function of the untrapped particles only. In (14) we can add the distribution function of the particles trapped in potential holes. We consider that in the holes the distribution function has a plateau.
It is more convenient to proceed to the reference frame where the wave is at rest. In this frame, expression (14) takes the form

$$f = \frac{n_0}{\sqrt{\pi} v_T} \exp(-\left(v_{ph} \mp \sqrt{v^2 - \frac{2e}{m}(\varphi + \varphi_0)} \right)^2 / v_T^2) \quad \text{for } v < -v_1 \text{ and } v > v_1 \text{ (untrapped particles)} \tag{15}$$

$$f = \frac{n_0}{\sqrt{\pi} v_T} \exp(-v_{ph}^2 / v_T^2) \quad \text{for } |v| < v_1 \text{ (trapped particles), where } v_1 = \sqrt{\frac{2e}{m}(\varphi + \varphi_0)} \tag{16}$$

We examine the electric charge density associated with the plateau.

The charge density is equal to $\rho = e\left(n_0 - \int_{-\infty}^{\infty} f dv \right)$ (17)

Then we find the charge density of the secondary waves, by substituting the distribution function for the trapped particles from (16) to (17)

$$\rho = e(n_0 - \frac{2 n_0 v_1}{\sqrt{\pi} v_T \exp(v_{ph}^2 / v_T^2)}) \tag{18}$$

The charge density can be presented as a Fourier series

$$\rho = \sum_{n=1}^{\infty} a_n \cos nkx + b_n \sin nkx\ , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \rho \cos(nkx) d(kx)\ , \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \rho \sin(nkx) d(kx) \tag{19}$$

where a_n and b_n are the coefficients of the Fourier series. Along with the charge of the primary wave (n = 1), there are charges referred to higher harmonics (n = 2, 3, 4). The basic contribution to the creation of higher harmonics is due to the resonance electrons, which velocity is close to the phase velocity of the wave. In formula (16), written for the reference frame of the wave, these are particles with the velocities close to zero. The higher harmonics in the charge distribution will result in the generation of higher-frequency waves.
Thus we obtain the coefficients for the *n-th* harmonics

$$a_n = \frac{16 n_0 \sqrt{\frac{e^3 \varphi_0}{m}}}{\pi \sqrt{\pi} v_T \exp(v_{ph}^2 / v_T^2)} * \frac{(-1)^n}{1 - 4n^2}\ ; \qquad b_n = 0$$

The charge density of the wave with n harmonics is

$$\rho_n = \frac{16\, n_0 \sqrt{\frac{e^3 \varphi_0}{m}}}{\pi \sqrt{\pi}\ v_T \exp(v_{ph}^2 / v_T^2)} * \frac{(-1)^n}{1 - 4\, n^2} \cos nkx. \tag{20}$$

To find the amplitude of the electric field of the secondary waves, we use one of the Maxwell equations:
div**E**=4πρ (21)
The problem we consider is one-dimensional. Then, equations (21) take the form:

$$\frac{\partial E}{\partial x} = 4\pi\rho$$

Then, we express the electric field E:

$$E = \int 4\pi\rho dx = \int \sum_{n=1}^{\infty} \frac{64\, n_0 \sqrt{\frac{e^3\varphi_0}{m}}}{\sqrt{\pi}\, v_T \exp(v_{ph}^2 / v_T^2)} * \frac{(-1)^n}{1-4\,n^2} \cos nkxdx = \sum_{n=1}^{\infty} \frac{64\, n_0 \sqrt{\frac{e^3\varphi_0}{m}}}{k\sqrt{\pi}\, v_T \exp(v_{ph}^2 / v_T^2)} * \frac{(-1)^n}{n-4\,n^3} \sin nkx.$$

From (20) we can see that the amplitude of electric charge density strongly subsides for n> 2, therefore, we shall restrict our consideration by the second harmonics only.

The charge density of the secondary wave is

$$\rho_2 = \frac{16 n_0 \sqrt{\frac{e^3\varphi_0}{m}}}{15\pi\sqrt{\pi} v_T \exp(v_{ph}^2 / v_T^2)} \cos 2kx$$

The electric field amplitude of the second wave is

$$E_2 = \frac{2{,}133\, n_0 \sqrt{\frac{e^3\varphi_0}{m}}}{k\sqrt{\pi}\, v_T \exp(v_{ph}^2 / v_T^2)} \qquad (22)$$

We estimate the density of the wave energy, given to the electrons as a result of reflection in the wave potential hole, by equation (22). We find only the amplitude of the electric field, therefore *sin2k* can be rejected:

$$W_{w2} = \frac{E_2^2}{8\pi} = \frac{0.568 n_0^2 e^2 v^2}{k^2 \pi^2 v_T^2 \exp(v_{ph}^2 / v_T^2)} \qquad (23)$$

The density of the energy spent for plateau formation is equal to [6]:

$$\Delta E = m\left(\sqrt{\frac{e\varphi_0}{m}}\right)^3 v \frac{\partial f_0}{\partial v}\Big|_{v=\frac{\omega}{k}} \qquad (24)$$

Then for Maxwellian distribution function (9) we get

$$\Delta E = -\frac{2 n_0 m v^5}{\sqrt{\pi} v_T^3} \exp(-v_{ph}^2 / v_T^2) \qquad (25)$$

Let us compare the density of energy spent on the formation of secondary waves (23) and that spent on the formation of plateau (25) for the parameters taken from [3]: $n_0 = 10^{20}\, m^{-3}$, $\lambda = 2{,}3*10^{-2} m$, $v_T = 10^3\, m/s$.

$$-\frac{2mv^3}{\sqrt{\pi} v_t} : \frac{0{,}568 n_0 e^2}{k^2\pi^2}$$

$$\frac{10{,}293 v^3}{v_T} : \frac{0{,}147*10^{-7} n_0}{k^2} = v^3 : 5{,}955*10^{11}$$

At the velocities of about v_T,the energy exchange between the particles and wave will not occur, the wave will fade in a half-period of oscillations. At the velocities larger than v_T, the estimations show that the process of plasma wave damping at the expense of generation of secondary waves noticeably contributes to the general process of damping at the formation of the plateau. Therefore, this mechanism appears to be effective.

Conclusions

The idea of non-linear damping of plasma waves, at the expense of generation of secondary waves at the oscillations of the resonance electrons in a potential hole of the wave, is proposed. The characteristics of the secondary waves, such as the charge density and electrical field amplitude, are calculated. The energy density for the second harmonics of the wave is also found, and the estimation of the efficiency of the mechanism is performed. It is shown that the damping at the expense of secondary wave generation noticeably contributes to the general process of damping at the formation of the plateau. Thus, the mechanism we propose appears to be effective.

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