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BACKWARD WAVE OSCILLATOR REGIME OF WHISTLER CYCLOTRON INSTABILITY IN AN INHOMOGENEOUS MAGNETIC FIELD

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Abstract

We present the linear theory for the backward wave oscillator generation regime of whistler waves in the Earth's magnetosphere. Using a parabolic profile of the magnetic field and a linear expression for the resonant current in the case of a zero order distribution function with a step discontinuity in the velocity component parallel to the magnetic field, we investigate the modes of the system by means of a search procedure. The existence of at least one mode exponentially growing in time is indicative of absolute instability, and such modes have been found. Therefore, earlier prediction of such a regime, based on the homogeneous magnetic field model, is confirmed. The dependence of growth rates on the frequency mismatch and energetic electron density has been studied. These results yield the characteristic spatial profile and temporal growth rate of small-amplitude whistler-wave disturbances, which are likely to be the seeds for chorus emissions.

Introduction

It is known that quasi-linear interactions of energetic electrons with noise-like emissions having an upper frequency cutoff lead to formation of a step-like feature on the electron distribution function in geomagnetic field-aligned velocities (*Trakhtengerts et al.*, 1986; *Nunn and Sazhin*, 1991; *Trakhtengerts et al.*, 1996). *Trakhtengerts* (1995) suggested that such a velocity distribution provides the backward wave oscillator (BWO) generation regime in the magnetospheric cyclotron maser, i.e., the absolute instability of a quasimonochromatic whistler wave in a near-equatorial region of the Earth's magnetosphere, and studied its properties in the approximation of a homogeneous magnetic field. BWO regime has been suggested by *Trakhtengerts* (1995, 1999) as a mechanism for chorus generation.

In this paper, we prove that the absolute instability leading to the BWO generation regime is also realized in an inhomogeneous magnetic field. For that, we solve the linearized self-consistent equations for the wave amplitude and energetic-electron resonant current in a parabolic external magnetic field. On this basis, we obtain the spatial profile of an absolutely unstable whistler-mode wave formed due to cyclotron resonant interactions and calculate the growth rate of the absolute instability as function of the energetic-electron density and frequency.

Basic equations

We start from the equations for the cyclotron resonant interaction of a quasi-monochromatic whistler wave and a population of energetic electrons. The equation for the slowly varying amplitude of the wave magnetic field has the form (*Karpman et al.*, 1974; *Nunn*, 1974; *Omura et al.*, 1991)

$$\frac{\partial A}{\partial t} + v_{\mathrm{g}} \frac{\partial A}{\partial z} = \alpha_1 J \qquad (1)$$

where

$$\alpha_1 = -\frac{2\pi v_{\mathrm{g}}}{c N_{\mathrm{w}}} \qquad (2)$$

$N_{\mathrm{w}} = k\,c / \omega$ is the whistler wave refractive index, and J is the resonant current:

$$J = -e \int \frac{\left(\vec{v}\vec{A}^*\right)}{|A|} F \,\mathrm{d}^3 v \qquad (3)$$

The electron distribution function F can be obtained using the method of characteristics for the Vlasov kinetic equation. We assume that the unperturbed distribution F_0 has a step-like feature in the equatorial parallel velocities $v_{\parallel L}$:

$$F_0 = n_{\mathrm{b}} \left[1 + p\Theta\left(V_* + v_{\parallel L}\right)\right] f\left(v_{\parallel L}, I\right) \qquad (4)$$

where $\Theta(x)$ is the Heaviside unit function, $f(v_{\parallel L}, I)$ is the smooth part of the distribution, $I = v_\perp^2 / \omega_H$ is the 1st adiabatic invariant, ω_H is the electron gyrofrequency, p is the relative amplitude of the step, and $V_* > 0$ is the absolute value of the parallel velocity on the step. The subscript "L" denotes the values in the equatorial plane.

If we take into account only the step contribution to the resonant current and assume the medium to be weakly inhomogeneous, then the linearized kinetic equation yields the following differential equation for the resonant current J (see, e.g., *Trakhtengerts et al.*, 1999):

$$\frac{\partial J}{\partial t}+v_*\frac{\partial J}{\partial z}=i\Delta J+\alpha_2 A \qquad (5)$$

where $v_* = [V_*^2 - I(\omega_H - \omega_{HL})]^{1/2}$ is the parallel velocity at some distance from the equator,

$$\Delta = \omega - \omega_H - kv_* \qquad (6)$$

is the mismatch from the cyclotron resonance,

$$\alpha_2 \approx \frac{e^2 pn_b}{2m}\frac{I_0\omega_{HL}}{V_*^2}\left(\frac{\omega_{HL}}{\omega}-1\right) \qquad (7)$$

and I_0 is the characteristic value of the 1st adiabatic invariant.

Equations (1) and (5) determine the spatio-temporal evolution of a small-amplitude whistler wave at the linear stage of the instability in the presence of the stepped distribution function. Our goal is to show that such a wave-particle system can be absolutely unstable.

The absolute instability means that there exists a solution growing temporally at a given point inside the domain of definition. For a spatially bounded system, it also means that the output signal grows from the noise level in the absence of an input signal. Therefore, we should find a solution which has zero amplitude A at the input and grows exponentially in time at the output. Note that the boundaries of the system are not strictly determined in the case of a magnetospheric maser, so we place them at arbitrary points far enough from the equator and check that their position does not influence the final solution. As we show below, the effective system length is determined by the geomagnetic field inhomogeneity. Hereafter, we assume parabolic z-dependence of the magnetic field:

$$H = H_L\left(1+z^2/a^2\right), \qquad (8)$$

where $a = (2^{1/2}/3)\,R_0\,L$, and R_0 is the Earth radius.

It can be seen from (8) that the phase mismatch due to the magnetic field inhomogeneity becomes of order unity at a length

$$\Delta z \sim l_{\rm BWO} \equiv a\left(\frac{V_*}{a\omega_{HL}}\right)^{1/3} \sim a(ka)^{-1/3}, \qquad (9)$$

For whistler waves in the Earth's magnetosphere, $l_{\rm BWO}/a \sim 10^{-3} \ll 1$. This means that at distances important for the BWO regime (i.e., several times $l_{\rm BWO}$ from the equator), we can neglect the variation in any coefficient entering the BWO system of equations, except the mismatch Δ.

It is convenient to normalize the spatial variable z to $l_{\rm BWO}$ and time t, to the time of flight $t_{\rm BWO} = l_{\rm BWO}/V_*$ of resonant electrons through the distance $l_{\rm BWO}$:

$$\xi = z/l_{\rm BWO}\,, \quad \tau = t/t_{\rm BWO} \qquad (10)$$

For analyzing the linear stability properties, solutions of the form $(A, J) \propto \exp(i\hat{\Omega}t)$ are of relevance ($\hat{\Omega} = \Omega - i\gamma$ is the slow complex frequency). Below, without loss of generality, we assume $\Omega = 0$, since a real addition to the wave frequency is equivalent to a change in the mismatch Δ. In this case, excluding J from equations (1) and (5), one obtains the single 2-nd order equation

$$A'' + i\delta A' + qA = 0\,, \qquad (11)$$

where

$$\delta = \Delta\cdot t_{\rm BWO}\,, \quad q = \frac{\alpha_1\alpha_2}{v_g V_*}l_{\rm BWO}^2 \qquad (12)$$

and primes denote differentiation over ξ. For the parabolic magnetic field (8),

$$\delta \cong \delta_L - C\xi^2\,, \qquad (13)$$

where $C = 1 + \tilde{\omega} + (1-\tilde{\omega})(I_0\,\omega_{HL}/V_*^2)$ is the numerical factor of order unity, and $\tilde{\omega} \equiv \omega/\omega_{HL}$ is the normalized wave frequency.

The appropriate boundary conditions for the BWO equations (1) and (5) represent the absence of the resonant current and a finite output wave amplitude at the right-hand boundary of the interaction region:

$$\begin{aligned} A\big|_{z=+l} &= A_{\rm out} \\ J\big|_{z=+l} &= 0\,. \end{aligned} \qquad (14)$$

Using Eq. (1), the second of these equations is rewritten in terms of the wave amplitude as

$$\left(A^{-1}{\rm d}A/{\rm d}z\right)_{z=+l} = i\gamma/v_g\,. \qquad (15)$$

We remind that the boundaries are specified here at arbitrary points at the opposite sides from the equator, far enough compared with the characteristic scale length $l_{\rm BWO}$. This makes a significant difference from the case of a homogeneous magnetic field (*Trakhtengerts*, 1995), where the externally imposed boundaries determine the scale length and, hence, the instability parameters.

As was noted above, an absolutely unstable solution of the system (1) and (5) or the equivalent BWO equation (11) starts from zero at some point inside the domain of definition and takes a finite value at the output. Such a solution exists only for a certain, if any, combination of parameters. To find this solution and the corresponding parameters, including the growth rate, we integrate Eq. (11) backward in ξ from $\xi = \xi_m = +l/l_{\rm BWO}$ ($z = l$). This procedure is repeated for different values of the growth rate γ and equatorial frequency mismatch δ_L, until the amplitude reaches zero somewhere inside the integration region. Reaching this condition means that we have found the spatial profile of the unstable wave and the corresponding growth rate.

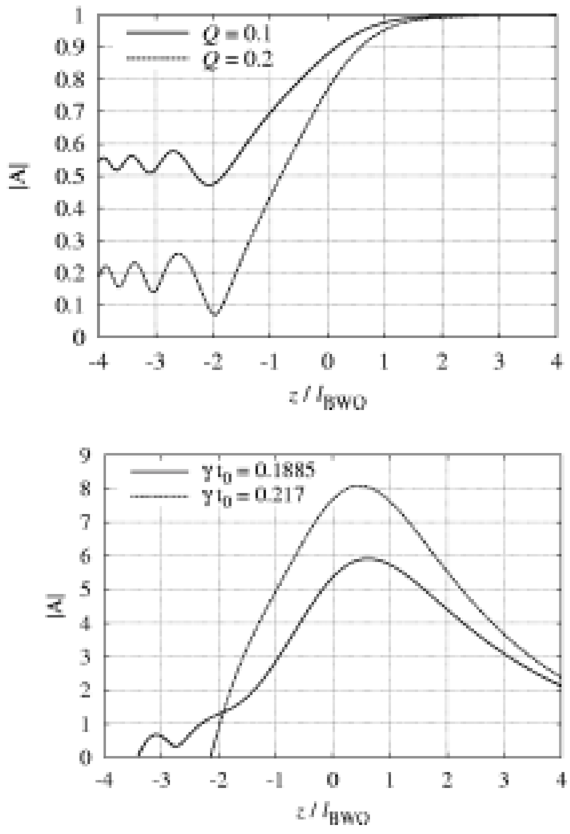

Figure 1. Spatial dependences of the wave amplitude in BWO. *Upper panel*: stable solutions ($\gamma = 0$) for the mismatch $\delta_L = 0.75$ and two values of the normalized beam density q; *Lower panel*: unstable solutions for $q = 0.5$, $\delta_L = 0.75$, and two values of γ satisfying the dispersion relation.

Results

Figure 1 shows spatial profiles of the whistler-wave amplitude in BWO. The upper panel represents the case of stable convective amplification ($\gamma = 0$), while the lower panel shows exponentially growing solutions obtained using the procedure described in the previous section. They correspond to the same value of the normalized mismatch δ_L and different values of the normalized growth rate γ. In accordance with the above discussion, they start from zero amplitude at some distance z_0 to the left from the equator and have an exponential spatial decay at large $\xi = z / l_{\mathrm{BWO}}$.

These solutions are actually two unstable "eigenmodes" of BWO. They can be regarded as lowest eigenmodes, since they have the smallest numbers of the amplitude spatial maxima. Note that different modes are localized in different space regions (i.e., have different extent to negative z). This makes a significant difference from a usual boundary-value problem, in which the boundaries determine the domain of definition for all modes. In our problem, each mode has its own domain of definition.

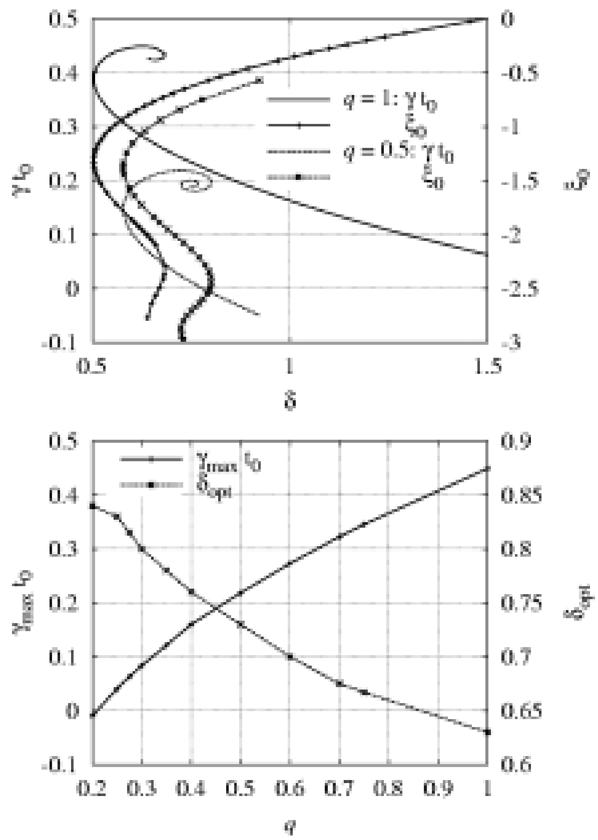

Figure 2. *Upper panel*: The growth rate of the absolute instability in the BWO as a function of the mismatch δ_L at $\omega/\omega_{HL} = 0.25$ and two values of q=1 and 0.5. The coordinate of the point ξ_0 at which the amplitude goes to zero is also plotted. *Lower panel*: The maximum growth rate γ and the corresponding optimum mismatch δ_{opt} as function of the normalized "beam" electron density q ($\omega/\omega_{HL} = 0.25$).

Figure 2 (upper plot) shows the dependence of the growth rate on the frequency mismatch δ_L for two fixed values of q. The left boundaries ξ_0 for each solution is shown here too. It is seen that the frequency range of the absolute instability is limited from both above and below: there is no solution of eigenmode type at $\delta_L < 0.5$ (note that this lower limit almost does not depend on q), while such solutions are damped ($\gamma < 0$) if δ_L exceeds some upper limit which increases with q. For $q \leq 0.2$, the instability range vanishes, i.e., the latter inequality represents the threshold of the BWO regime.

The lower panel in Fig. 2 shows the growth rate maximized over δ_L and the corresponding optimum value δ_{opt} as functions of q. Here, it is clearly seen that the BWO instability occurs if $q \geq 0.2$.

Figure 3 shows snapshots of the wave amplitudes in BWO obtained by direct numerical integration of the spatio-temporal equations (1) and (5) to check the above stability analysis. It is verified that the unstable solutions grow exponentially in time, while the spatial structure is preserved.

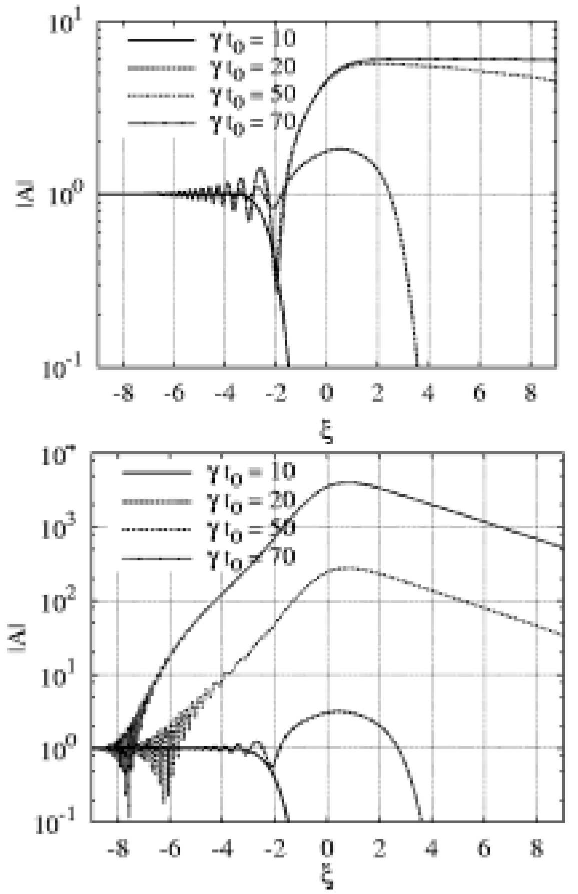

Figure 3. Spatial profiles of whistler wave amplitude at several subsequent times for the marginally stable case q = 0.2 and unstable case q = 0.4. δ_L = 0.75 and ω/ω_{HL} = 0.25.

Discussion and conclusions

The main result of this paper is the proof of the fact that the absolute instability of whistler waves (BWO regime) in the presence of a step-like feature on the electron distribution function exists in an effectively unbounded system with an inhomogeneous (parabolic) magnetic field. In this case, the wave structure is determined by the magnetic field inhomogeneity which dictates the spatial scale of the unstable waves. Using numerical calculations, we have obtained the spatial structure of the modes exponentially growing in time and the dependence of the growth rate on the plasma and wave parameters.

This study confirms the idea of *Trakhtengerts* (1995) that the BWO regime in an inhomogeneous magnetic field is similar to that in a homogeneous medium if the device length is replaced by the characteristic scale length l_{BWO}. Therefore, the numerical estimates of the cited paper are also confirmed. We conclude that the BWO regime is indeed a good candidate to explain chorus generation in the Earth's magnetosphere (*Trakhtengerts*, 1999).

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References

Karpman, V. I., Y. N. Istomin, and D. R. Shklyar, Nonlinear theory of a quasimonochromatic whistler mode packet in inhomogeneous plasma, *Plasma Phys.*, **16**, 685–703, 1974.

Nunn, D., A self-consistent theory of triggered VLF emissions, *Planet. Space Sci.*, **22**, 349–378, 1974.

Nunn, D., and S. S. Sazhin, On the generation mechanism of hiss-triggered chorus, *Ann. Geophys.*, **9**, 603–613, 1991.

Omura, Y., D. Nunn, H. Matsumoto, and M. J. Rycroft, A review of observational, theoretical and numerical studies of VLF triggered emissions, *J. Atmos. Terr. Phys.*, **53**, 351–368, 1991.

Trakhtengerts, V. Y., Magnetosphere cyclotron maser: Backward wave oscillator generation regime, *J. Geophys. Res.*, **100**, 17205–17210, 1995.

Trakhtengerts, V. Y., A generation mechanism for chorus emission, *Ann. Geophys.*, **17**, 95–100, 1999.

Trakhtengerts, V. Y., V. R. Tagirov, and S. A. Chernous, Flow cyclotron maser and impulsive VLF emissions, *Geomagn. Aeron.*, **26**, 99–106, 1986.

Trakhtengerts, V. Y., M. J. Rycroft, and A. G. Demekhov, Interrelation of noise-like and discrete ELF/VLF emissions generated by cyclotron interactions, *J. Geophys. Res.*, **101**, 13293–13303, 1996.

Trakhtengerts, V. Y., Y. Hobara, A. G. Demekhov, and M. Hayakawa, Beam-plasma instability in inhomogeneous magnetic field and second order cyclotron resonance effects, *Phys. Plasmas*, **6**, 692–698, 1999.