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VISCOUS FRICTION IN LAMINAR AND TURBULENT HYDRODYNAMIC FLOWS

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Abstract. An attempt is made to generalize the well-known Newtonian law of viscous friction which establishes a linear relationship between viscous stresses and flow velocity gradients and is in a good agreement with experimental data only for the flows with small velocity gradients. A new general approach is suggested which allows us to establish a non-linear relationship between viscous stresses and flow velocity gradients. The suggested approach is based on the application of the experimental data on the fluid flowing in circular pipes of constant cross section. Using the suggested approach for water, we have derived the generalized Newtonian law of viscous friction which is valid for flows with arbitrary values of velocity gradients.

1. Introduction

For description of the dynamical behaviour of gases and liquids, the Navier-Stokes equations are widely used that are valid for Newtonian liquids in which the relation between viscous stresses and flow velocity gradients is described by the Newtonian law of viscous friction

$$\hat{\tau} = 2\,\mu\,\hat{\mathcal{E}} \tag{1}$$

where $\hat{\tau}$ is the viscous stress tensor, μ is the coefficient of viscosity, which is independent of the flow velocity, and $\hat{\mathcal{E}}$ is the tensor with the components which may be written in Cartesian coordinates in the following form:

$$\mathcal{E}_{i\,j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\,\delta_{i\,j}\,\mathrm{div}\,\vec{v}\right) \tag{2}$$

where $\vec{v}$ is the vector of the flow velocity with the components in Cartesian coordinates v_1, v_2, and v_3, and $\delta_{i\,j}$ is the Kronecker delta [*Batchelor, 1970; Germain, 1973; Schlichting, 1979*].

The relation between the viscous stress tensor components and those of flow velocity gradients is linear in Newtonian liquid. Therefore, under the condition of the flow velocity gradients increasing unlimitedly, the viscous stresses ought to increase infinitely, too. However, this contradicts the results of measurements of viscous drag on rigid bodies. It is experimentally established that the resistance of rigid bodies always stays finite even at very high Reynolds numbers when a flow velocity gradient tends to infinity on a boundary of a body [*Batchelor, 1970; Schlichting, 1979*]. Hence, the model of a Newtonian liquid is not applicable for description of the dynamical behaviour of liquids and gases for the flows with large velocity gradients whereas it provides a good agreement with experimental data for the flows with small velocity gradients.

The purpose of this paper is to establish a non-linear relationship between viscous stresses and flow velocity gradients that would be valid for the flows with arbitrary values of velocity gradients, in particular, with those tending to infinity. It can be noted that the latter condition is sufficient for transition from laminar to turbulent flowing. As a consequence, the flows with high velocity gradients are usually turbulent [*Schlichting, 1959; Zhigulev and Tumin, 1987*].

2. Problem formulation

To establish the law of viscous friction or so-called rheological equation of state appropriate for the flows with arbitrary large velocity gradients, we use the phenomenological hydrodynamic approach. The rheological equation of state has been chosen among generalized Newtonian rheological models [*Astarita and Marrucci, 1974*]. The latter is a natural generalization of ordinary Newtonian law of viscous friction (1) and may be written in the following form:

$$\hat{\tau} = 2\,\eta(S)\,\hat{\mathcal{E}} \tag{3}$$

where $\eta(S)$ is the so-called apparent viscosimetric viscosity (AVV) and $\hat{\mathcal{E}}$ is the tensor defined in (2). The AVV is assumed to be a continuous non-negative function of the parameter S connected with the flow velocity by the formula

$$S = 2\sum_{i,\,j=1}^{3}(\mathcal{E}_{i\,j})^2 = \frac{1}{2}\sum_{i,\,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 - \frac{2}{3}\,(\mathrm{div}\,\vec{v})^2\,. \tag{4}$$

The AVV has the same dimentions as the viscosity coefficient and must satisfy the following conditions:

$$\eta(0) = \mu\,, \qquad \left.\frac{d^k\eta(S)}{dS^k}\right|_{S=0} = 0 \qquad k = 1, \ldots, n \tag{5}$$

where $n \geq 1$ is a certain integer number. Without any restrictions, the AVV may be represented in the following form :

$$\eta(S) = \mu\,\Phi(S\mu^2/{\tau_m}^2) \tag{6}$$

where $\Phi(x)$ is a continuos non–negative dimensionless function and τ_m is a certain constant having the same dimentions as the stress. In general, the function $\Phi(x)$ may be represented by the formula,

$$\Phi(x) = \psi(\sqrt{x})/\sqrt{x}\,, \tag{7}$$

where $\psi(x)$ is a continuous non–negative dimensionless function, too. Therefore, the AVV can be expressed by the relation

$$\eta(S) = \frac{\tau_m}{\sqrt{S}}\,\psi\left(\frac{\mu}{\tau_m}\sqrt{S}\right)\,. \tag{8}$$

It is obvious that for ordinary Newtonian rheological model (1) the functions $\eta(S)$, $\Phi(x)$, and $\psi(x)$ may be expressed by the relations

$$\eta(S) = \mu\,, \quad \Phi(x) = 1\,, \quad \psi(x) = x\,. \tag{9}$$

For generalized Newtonian rheological models (3), the accurate description of behaviour of the AVV, $\eta(S)$ and auxiliary functions $\Phi(x)$ and $\psi(x)$ is a problem of principal interest. In the present paper as an initial step, we are interested in deriving particular expressions for the functions $\eta(S)$, $\Phi(x)$, and $\psi(x)$ only for incompressible liquid.

3. Theoretical preliminaries

We consider a stationary, one–dimensional, isothermal, viscous flow of incompressible liquid in a pipe of a circular cross section. Generalized Newtonian rheological model (3) is supposed to be valid for the liquid. Using relation (3), we can write the generalized Navier–Stokes equations [*Batchelor, 1970*] in cylindrical coordinates (r, φ, z), with the z -axis coinciding with the central line of a pipe and pointing in the direction of the motion. As the flow under consideration is axisymmetrical and one–dimensional the velocity components in the r, φ, and z directions (v_r, v_φ, and v_z, respectively) must satisfy the following conditions :

$$v_r = v_\varphi = 0\,, \quad v_z = v(r)\,. \tag{10}$$

Hence, the generalized Navier–Stokes equations are reduced to a single equation of the form

$$\frac{d(r\,\tau_{rz})}{dr} = \frac{dp}{dz}\cdot r \tag{11}$$

where τ_{rz} is the component of the viscous stress tensor, defined in (3), and p is the pressure. In the flow under consideration, it turns out that

$$\mathcal{E}_{rz} = \frac{1}{2}\,\frac{dv(r)}{dr}\,, \quad S = \left(\frac{dv(r)}{dr}\right)^2\,. \tag{12}$$

Moreover, in any circular pipe flow the radial profile of the longitudinal component of the flow velocity $v(r)$ was experimentally established to be always a convex function for which the following conditions must be satisfied :

$$\frac{d^2v(r)}{dr^2} < 0\,, \quad \frac{dv(r)}{dr} < 0 \quad \text{when} \quad r \in (0, R]\,; \quad \frac{dv(0)}{dr} = 0 \tag{13}$$

where R is the radius of the pipe. Integrating Eq. (11) from 0 to r and taking into account (3), (8), (12), and (13) we get

$$\psi\left(-\frac{\mu}{\tau_m}\,\frac{dv(r)}{dr}\right) = -\frac{dp}{dz}\,\frac{r}{2\,\tau_m} \quad \text{when} \quad r \in [0, R] \tag{14}$$

which is the basic relation for the further investigation.

4. Formulation of the new approach

It is generally understood now that a flow at a high Reynolds number may be divided into two motions : first, the averaged flow characterized by the velocity changing smoothly ; second, the pulsation flow with the mean velocity equal to zero. The amplitude of the velocity pulsation is much smaller than the averaged velocity of the fluid. A laminar flow relates only to the former motion. We attempt to describe the viscous friction in both laminar and turbulent flows by a unified approach using generalized Newtonian rheological model (3) and the generalized Navier–Stokes equations, in which (3) is used, only with respect to the averaged velocity of the fluid. In the approach under consideration, the AVV is the parameter of principal interest. To derive a particular expression for the AVV it is convenient to use the auxiliary function $\psi(x)$ which satisfies relation (14) and is connected with the AVV by expretion (8). The right–hand side of relation (14) contains the pressure gradient, $\frac{dp}{dz}$. We suggest that in an arbitrary circular pipe flow the pressure gradient may be represented as

$$\frac{dp}{dz} = -\frac{\triangle P_0}{L} + \left(\frac{dp}{dz}\right)_a \tag{15}$$

where $\triangle P_0$ is the pressure difference and L is the length of the pipe, which can be measured experimentally, and $\left(\frac{dp}{dz}\right)_a$ is the additional pressure gradient originated from the combined effects of a pulsation flow and surface roughness. It is known that the effect of roughness is negligible at low Reynolds numbers when the pulsations are absent and, hence, expression (15) is reduced to

$$\frac{dp}{dz} = -\frac{\triangle P_0}{L}\,. \tag{16}$$

It is known that the averaged motion in an abitrary circular pipe flow is one–dimensional. Supposing that the averaged motion obeys relation (16), equation (14) can be written as

$$\psi\left(-\frac{\mu}{\tau_m}\frac{dv(r)}{dr}\right) = \frac{\triangle P_0}{2\,L\,\tau_m}\,r \qquad r \in [\,0, R\,]\,. \tag{17}$$

It can be seen that, as a consequence of conditions (13), the argument of the function ψ in equation (17) is a non–increasing function of r, for which the equality $\psi(0) = 0$ is true. So the right–hand side of equation (17) is a ramp function of r. Hence, the function $\psi(x)$ must be monotonic. As a consequence, the function $\psi(x)$ must have an inverse function which is denoted by $\psi_*(x)$ and determined in the range $[\,0, \triangle P_0\,R/(2\,L\,\tau_m)\,]$. Evidently, the function $\psi_*(x)$ is monotonic, too. Using the inverse function, equation (17) can be rewritten as

$$-\frac{\mu}{\tau_m}\frac{dv(r)}{dr} = \psi_*\left(\frac{\triangle P_0\,r}{2\,L\,\tau_m}\right) \qquad r \in [\,0, R\,]\,. \tag{18}$$

The derived relation allows us to obtain approximately the function $\psi_*(x)$ by using accurate experimental data on the radial profiles of the longitudinal component of the averaged velocity, $v(r)$, in a circular pipe flow for an arbitrary incompressible fluid. The AVV may be easily obtained by transferring from $\psi_*(x)$ to $\psi(x)$ and using relation (8). We have briefly described the main idea of the new approach which allows us to establish a non–linear behaviour of the AVV. More detailed description of the suggested approach may be found in the study by Mingalev et al. [1997].

5. Application of the new approach to water

Up to now there have been very many experimental studies of water flowing in a pipe of a circular cross section in the low and high Reynolds number limit. The review of these works and many details of circular pipe flowing may be found in the study by Schlichting [1979]. In particular, this study includes the experimental resuslts obtained earlier by other authors. The analysis of these results, with regards to their application to the derivation of the function

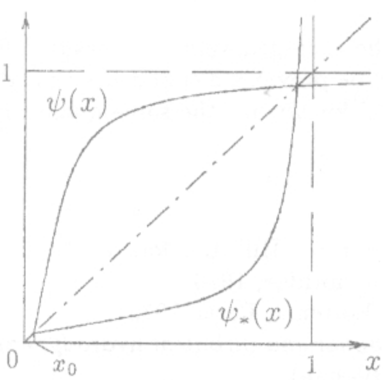

Fig. 1. The qualitative diagrams of the functions $\psi(x)$ and $\psi_*(x)$ (solid lines). The result for ordinary Newtonian rheological model (1) is shown by the dashed line.

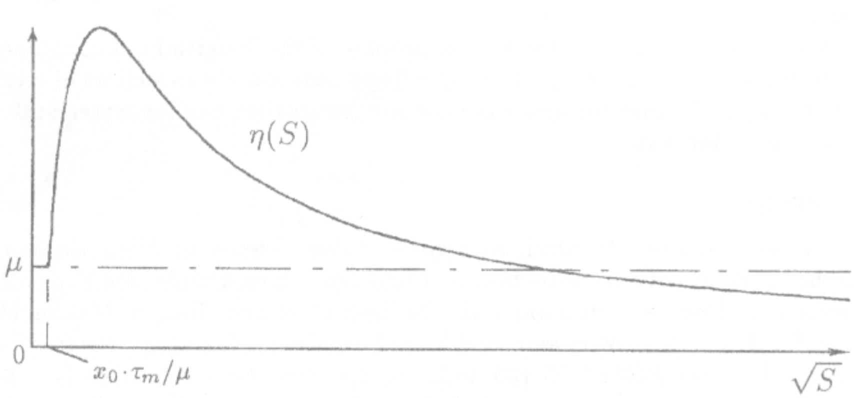

Fig. 2. The qualitative diagram of the apparent viscosimetric viscosity (AVV), $\eta(S)$. The result for ordinary Newtonian rheological model (1) is shown by the dashed line.

$\psi_*(x)$ (cf. (18)), has been performed by Mingalev et al. [1997]. From this study, it follows that the only possible function $\psi_*(x)$ may be obtained by using equation (18) and the experimental data on the circular pipe flowing. The qualitative behaviour of the functions $\psi_*(x)$ and $\psi(x)$ obtained by using the suggested approach for water, is shown in Fig. 1. The qualitative behaviour of the AVV is shown in Fig. 2. Generalized Newtonian rheological model (3) using the AVV in the form presented in Fig. 2 provides finite magnitudes of the viscous stresses even under the condition that the flow velocity gradients tend to infinity. It turns out that the parameter τ_m introduced in relation (6) and presented in expression (8) has the physical meaning of the most possible viscous stress in the fluid [Mingalev et al., 1997].

6. The quantitative model for water

It shoud be noted once more that the suggested approach allows us to get the form of the AVV only appoximately. However, it is more convenient to use an analytical representation of the AVV for numerical analysis of fluid flows. By using relation (8), the AVV, $\eta(S)$, may be expressed in terms of the function $\psi(x)$. Thus, we give the analytical representation only for the function $\psi(x)$ which is

$$\psi(x) = \begin{cases} x + A_1\left(\frac{x}{x_0}\right)^{a_1} + A_2\left(\frac{x}{x_0}\right)^{a_2}\left(1-\frac{x}{x_0}\right)^2 & \text{for} \quad x \in [\,0\,,\, x_0\,)\,, \\ 0.041 + \sum_{m=1}^{6} B_m\left(\frac{x-x_0}{x_1-x_0}\right)^m & \text{for} \quad x \in [\,x_0\,,\, x_1\,)\,, \\ \frac{x^{1.1} - (0.065277)^{1.1} + (\,0.0008\,x^{-1.2} + 0.07\,x\,)\,e^{-100x^2}}{x^{1.1} + 0.003\,e^{-11x^2}} & \text{for} \quad x \in [\,x_1\,,\, +\infty\,) \end{cases} \tag{19}$$

where $x_0 = 0.034$, $x_1 = 0.1$, $A_1 = 0.007$, $a_1 = 22.990314$, $A_2 = -1.3603448$, $a_2 = 36$, $B_1 = 5.7333\,(x_1 - x_0)$, $B_2 = -9\,(x_1 - x_0)^2$, $B_3 = 16\,(x_1 - x_0)^3$, $B_4 = -0.11496$, $B_5 = 0.22296246$, $B_6 = -0.10784162$,

The analytical function $\psi(x)$ and its first and second derivatives are continuous. The accuracy of the analytical approximation given above is higher than 10^{-3} (0.1%). The analytical representation for the inverse function, $\psi_*(x)$, may be found in the study by Mingalev et al. [1997].

7. Concluding remarks

The new approach has been suggested which permits to get a particular expression for the generalized Newtonian law of viscous friction with the AVV being a point of primary interest. Due to the suggested approach, based on the application of the accurate experimental data on the radial profiles of the longitudinal component of the averaged velocity in a circular pipe flow, both the qualitative behaviour and analytical representation of the AVV have been established for water, which may be applied to analysis of both laminar and turbulent flows using the generalized Navier-Stokes equations. The expression found for the AVV of water includes two physical parameters: the well-known coefficient of viscosity μ and most possible viscous stress in a fluid τ_m, newly introduced. It may be expected that the established generalized Newtonian law of viscous friction is valid for arbitrary three-dimensional flows.

A great resemblance between the profiles of the longitudinal component of the averaged velocity measured for water and for air in the boundary layer flows near a plate as well as in a circular pipe provides a confidence that the law of viscous friction for gases is very similar to that one for water and the AVV for air has the same qualitative behaviour as for water.

References

Astarita, G., and G. Marrucci, *Principles of non-Newtonian Fluid Mechanics,* Mc Graw - Hill, London, 1974.

Batchelor, G. K., *An Introduction to Fluid Dynamics,* Cambridge Univ. Press., Cambridge, 1970.

Germain, P. *Cours de Mécanique des Milieux Continus, Tome I,* Masson et C[ie], Éditeurs, Paris, 1973.

Mingalev, I. V., O. V. Mingalev, and V. S. Mingalev, *A law of viscous friction for laminar and turbulent hydrodynamic flows, Preprint PGI-97-03-103,* Polar Geophysical Institute, Apatity, 1997 (in Russian).

Schlichting H., *Entstehung der Turbulenz, Handbuch der Phisik, Band VIII/1,* Berlin-Gottingen-Heidelberg, 1959.

Schlichting H., *Boundary Layer Theory,* Mc Graw-Hill, New York, 1979.

Zhigulev, V. N., and A. M. Tumin, *Transition to Turbulence,* Nauka, Novosibirsk, 1987 (in Russia).